Asymptotic Blowup Solutions in MHD Shell Model of Turbulence

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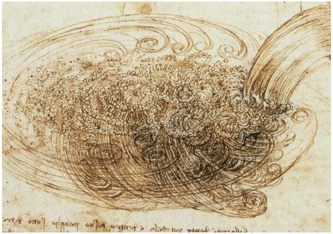
Turbulence

- Turbulent is how we characterize unpredictable and chaotic flow;
- Observable in a myriad of different natural phenomena;
  - Turbulent Mixing, present even in everyday tasks;
  - Atmospheric and maritime flow;
  - Solar weather;
- Important mechanism in processes involving energy and mass transport.



- Fluid dynamics is modeled after conservation and balance laws;
- Navier-Stokes equation for the incompressible flow,

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} &= -\frac{\nabla p}{\rho} + \nu \Delta \mathbf{v} + f, \\ \nabla \cdot \mathbf{v} &= 0, \end{aligned} \tag{1}$$

where the fluid density is represented by ρ, ν stands for the kinematic viscosity and f accounts for external forcing per unit of mass. Derived from the conservation of mass, the second equation is the condition for incompressibility.

Motivating question

- Problems:
  - A system of nonlinear differential equations with very rich behaviour, acting over an immense number of scales;
  - Are there solutions for every set of initial/boundary conditions? Are these solutions well defined for all time, i.e., **is there singularity formation in finite time (blowup)?**
- The existence of **blowup is and open problem** even in simple flow, such as 2D convective flow and 3D ideal flow.

The incompressible MHD equations

$$
\begin{aligned}
&\frac{\partial \mathbf{v}}{\partial t} - \nu \nabla^2 \mathbf{v} = -(\mathbf{v} \cdot \nabla)\mathbf{v} + (\mathbf{b} \cdot \nabla)\mathbf{b} - \nabla p, \\
&\frac{\partial \mathbf{b}}{\partial t} - \eta \nabla^2 \mathbf{b} = \nabla \times (\mathbf{v} \times \mathbf{b}), \\
&\nabla \cdot \mathbf{v} = 0 \quad , \quad \nabla \cdot \mathbf{b} = 0,
\end{aligned} \tag{2}
$$

where **v** and **b** are the velocity and induced magnetic fields, p is the (magnetic and kinetic) pressure; the density ρ has been taken as one. These equations follow from the Navier-Stokes equation taking into account the Lorentz force and from Maxwell equations.

- The nonlinear terms on the right-hand side redistribute magnetic and kinetic energy among the full range of scales of the system.
- Three-dimensional systems have three ideal quadratic invariants, the total energy (E), the total correlation (C) and total magnetic helicity (H) given as follows:

$$\begin{aligned} E &= \frac{1}{2}\int (\mathbf{v}^2 + \mathbf{b}^2) d^3x, \\ C &= \int \mathbf{v}\cdot\mathbf{b} d^3x, \\ H &= \int \mathbf{a}\cdot(\nabla\times\mathbf{a}) d^3x, \end{aligned} \tag{3}$$

where $\mathbf{a} = \nabla \times \mathbf{b}$.

Shell Models

- Discretization of the Fourier space onto concentric spherical shell, $k_{n-1} \leq \|\mathbf{k}\| < k_n$;
- The sequence $\{k_n\}_{n\in\mathbb{N}}$ is chosen as a geometric progression $k_n = k_0 h^n$; significantly reduces the degrees of freedom of the model;
- one or more scalar variables is assigned to each shell; these variables may account for fluid velocity, induced magnetic field, temperature deviation from its mean value, etc.
- The spectral Navier-Stokes equation can be written as

$$\frac{\partial v_j(\mathbf{k})}{\partial t} = -\, i \sum_{m,n} \int \left(\delta_{j,n} - \frac{k_j k'_n}{k^2} \right) v_m(\mathbf{k}') v_n(\mathbf{k} - \mathbf{k}') d^3\mathbf{k}' - \nu k^2 v_j(\mathbf{k}) + f_j(\mathbf{k}). \quad (4)$$

Following (4)'s structure, a shell model is defined

$$\frac{dv_n}{dt} = \mathcal{C}_n(v) - \mathcal{D}_n(v) + \mathcal{F}_n, \tag{5}$$

Imposing ideal conservation of Energy and Cross-correlation

$$E = \frac{1}{2}\sum(u_n^2 + b_n^2)\ , \qquad C = \sum u_n b_n$$

$$\begin{aligned} \frac{dv_n}{dt} &= k_n[\epsilon(v_{n-1}^2 - b_{n-1}^2) + v_{n-1}v_n - b_{n-1}b_n] \\ &\quad - k_{n+1}[v_{n+1}^2 - b_{n+1}^2 + \epsilon(v_n v_{n+1} - b_n b_{n+1})], \\ \frac{db_n}{dt} &= \epsilon k_{n+1}[v_{n+1}b_n - v_n b_{n+1}] + k_n[v_n b_{n-1} - v_{n-1}b_n]. \end{aligned} \tag{6}$$

where ν is the viscosity, η is the magnetic diffusivity, ϵ is an arbitrary coupling coefficient and $k_n = k_0 h^n$.

Blowup

- Example 1:

$$\frac{dy}{dt} = y^2. \tag{7}$$

Solutions of this equation have the form

$$y(t) = (t_c - t)^{-1} \to \infty \quad \text{as} \quad t \to t_c. \tag{8}$$

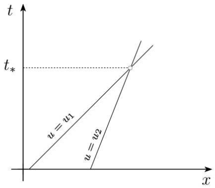
- Example 2: Cauchy problem for the inviscid Burgers equation

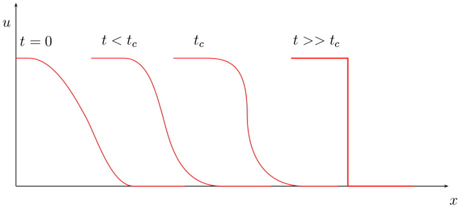
$$u_t + uu_x = 0 \tag{9}$$

solved using the characteristic curves

$$\frac{dt}{ds} = 1, \quad \frac{dx}{ds} = u. \tag{10}$$

- Solutions are constant along characteristics, which have slope $1/u_0$.
- If there are points $x_1 < x_2$ such that $u_0(x_1) > u_0(x_2)$,





then characteristic curves cross after some time. This leads to nonclassical (discontinuous) solution and divergence of the derivatives (blowup).

Blowup in our model

- We define the norms

$$\left\|v'\right\| = \left(\sum k_n^2 v_n^2\right)^{1/2}, \quad \left\|v'\right\|_\infty = \sup_n \, k_n \left|v_n\right| . \tag{11}$$

- Note that the norm $\|v'\|$ is then analogous to the enstrophy in fluid dynamics.
- Solutions of (6) are called regular (or classical) if

$$\left\|v'\right\| + \left\|b'\right\| < \infty \, . \tag{12}$$

Blowup criterion

- If the initial conditions at $t = 0$ satisfy the condition (12), there exists some $T > 0$ such that (6) has an unique regular solution $u(t)$ in the interval $[0, T)$.

Theorem

Let $v_n(t)$ and $b_n(t)$ be a smooth solution of (6) satisfying the condition (12) for $0 \leq t < t_c$, where t_c is the maximal time of existence for such solution. Then, either $t_c = \infty$ or

$$\int_0^{t_c} \left\| v' \right\|_\infty dt = \infty. \tag{13}$$

Proof schematics

- If (13) is true, then $\|v'\|_\infty$ is unbounded for $0 \le t < t_c$ and (12) is false; (13) is a sufficient condition for blowup.
- It is also a necessary condition; suppose there is blowup:
  - Evaluate $\frac{1}{2}\frac{d}{dt}\left(\|v'\|^2 + \|b'\|^2\right)$;
  - Manipulate each terms using the triangular, Cauchy-Schwarz and $k_n |v_n| \le \|v'\|_\infty$ inequalities, showing that there is a constant D such that

$$\frac{d}{dt}\left(\|v'\|^2 + \|b'\|^2\right) < D\,\|v'\|_\infty \left(\|v'\|^2 + \|b\|^2\right) \tag{14}$$

  - From the Gronwall inequality

$$\left(\|v'\|^2 + \|b'\|^2\right)_{t=t_c} < \left(\|v'\|^2 + \|b'\|^2\right)_{t=0} \exp\left(D\int_0^{t_c} \|v'\|_\infty \, dt\right) \tag{15}$$

Renormalization Scheme

Definition

Let τ be the renormalized time, implicitly defined by

$$t = \int_0^{\tau} \exp\left(-\int_0^{\tau'} R(\tau'')d\tau''\right) d\tau', \tag{16}$$

The renormalized velocity and magnetic variables are defined as

$$\begin{aligned} u_n &= \exp\left(-\int_0^{\tau} R(\tau')d\tau'\right) k_n v_n, \\ \beta_n &= \exp\left(-\int_0^{\tau} R(\tau')d\tau'\right) k_n b_n. \end{aligned} \tag{17}$$

Renormalized shell model

$$\frac{du_n}{d\tau} = -R(\tau)u_n + P_n\ , \qquad \frac{d\beta_n}{d\tau} = -R(\tau)\beta_n + Q_n \tag{18}$$

$$\begin{aligned} P_n &= \epsilon(h^2(u_{n-1}^2 - \beta_{n-1}^2) - u_n u_{n+1} + \beta_n \beta_{n+1}) \\ &\quad + h(u_{n-1}u_n - \beta_{n-1}\beta_n) - h^{-1}(u_{n+1}^2 - \beta_{n+1}^2), \\ Q_n &= \epsilon(u_{n+1}\beta_n - u_n\beta_{n+1}) + h(u_n\beta_{n-1} - u_{n-1}\beta_n). \end{aligned} \tag{19}$$

- $R(\tau)$ is found by imposing $\sum u_n^2 + \beta_n^2 = c$:

$$R(\tau) = \frac{\sum u_n P_n + \beta_n Q_n}{\sum u_n^2 + \beta_n^2} \tag{20}$$

- Moreover, at $t = \tau = 0$, $\sum u_n^2 = \sum k_n^2 v_n^2 = \|v\|^2 < \infty$. The same is true for $\sum \beta_n$, then $c < \infty$.

Lemma:

For any nontrivial initial conditions of finite ℓ^2-norm, a regular solution u_n and β_n of the renormalized system (18) exists and is unique for $0 \leq \tau < \infty$. This solution is related by (16) and (17) to the regular solution v_n and b_n of the original system (6) for $t < t_c$, where $t_c = \lim_{\tau \to \infty} t(\tau)$.

Proof schematics:

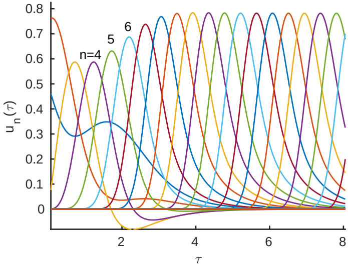
- As the renormalized model is constructed from (16) and (17), it is sufficient to show that (20) is well defined for all $0 \leq \tau$ corresponding to some $t < t_c$;
- Substituting P_n e Q_n and bounding each term (as in $\left|\epsilon h^2 u_{n-1}^2\right| \leq |\epsilon| \, h^2 c$), we conclude that $R(\tau) < \infty$ for all $\tau \geq 0$;
- As $|u_n| < c^{1/2}$ we see that $|k_n v_n| < \infty$, i.e. $\|v'\|_\infty < \infty$, for all t corresponding to a $0 \leq \tau < \infty$. From *Theorem* 1 it follows that $t < t_c$.

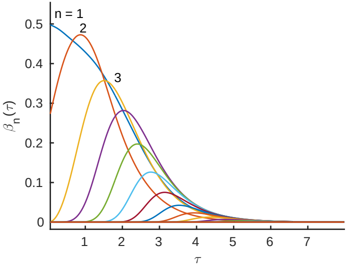
Symmetries

- The renormalized system has the following symmetries
  (S.R.1) $\tau \mapsto \tau/a$, $u_n \mapsto au_n$, $\beta_n \mapsto a\beta_n$ for arbitrary real constant a;
  (S.R.2) $\tau \mapsto \tau - \tau_0$ for arbitrary real τ_0;
  (S.R.3) $u_n \mapsto u_{n+1}$, $\beta_n \mapsto \beta_{n+1}$
- Lemma: From definition (17), the symmetries (S.R.1-3) lead to the following symmetries of the original shell model:
  (S.N.1) $t \mapsto t/a$, $v_n \mapsto av_n$, $b_n \mapsto ab_n$ for arbitrary real constant a;
  (S.N.2) $t \mapsto (t - t_0)/a$, $v_n \mapsto av_n$, $b_n \mapsto ab_n$, where a and t_0 are uniquely defined by τ_0 in (S.R.2);
  (S.N.3) $v_n \mapsto hv_{n+1}$, $b_n \mapsto hb_{n+1}$

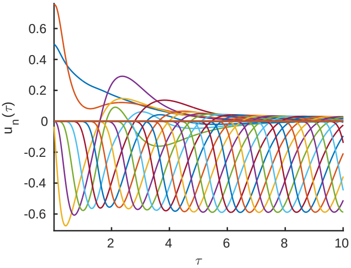
Types of solutions

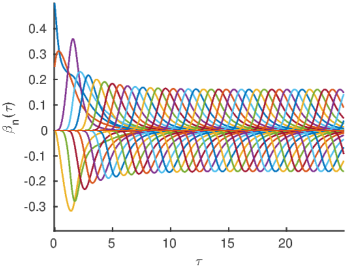
- We observe that, as the blowup time is taken to infinity, solutions develop as different types of waves travelling towards larger shells
- Travelling wave solutions



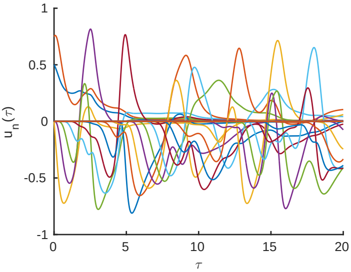


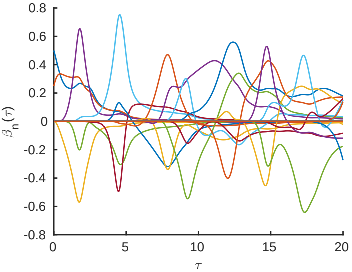
- Periodically pulsating wave solutions





- Chaotically pulsating wave solutions





Poincaré map

- We estimate the center n_w of a solution $w = (\dots, u_n, u_{n+1}, \dots, \beta_n, \beta_{n+1}, \dots)$ as

$$n_w(\tau) = \sum n(u_n^2 + \beta_n^2) / \sum (u_n^2 + \beta_n^2) \tag{21}$$

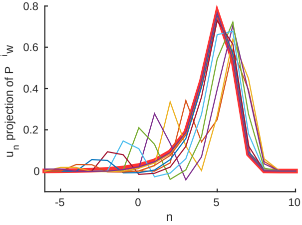
- We take the sequence τ_i as the times necessary for a solution center to travel by i shells

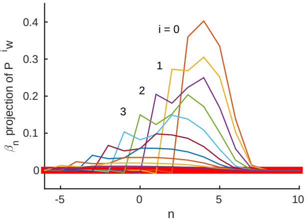
$$n_w(\tau_i) = n_w(0) + i \tag{22}$$

- We define a Poincaré map $\mathcal{P}$ as

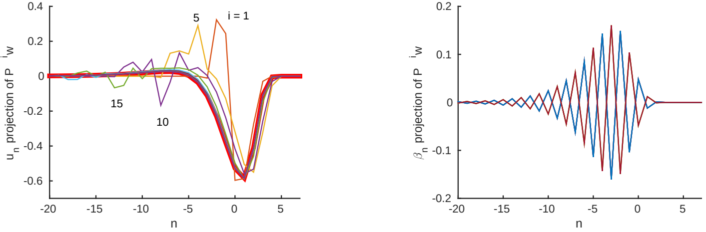
$$\begin{aligned} w'\mathcal{P} &= w, \quad u'_n = u_{n+1}(\tau_1), \quad \beta'_n = \beta_{n+1}(\tau_1), \\ w'\mathcal{P}^i &= w, \quad u'_n = u_{n+i}(\tau_i), \quad \beta'_n = \beta_{n+i}(\tau_i). \end{aligned} \tag{23}$$

Fixed-point attractor

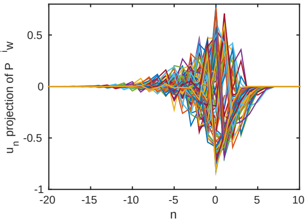


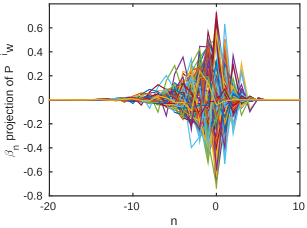


Periodic attractor

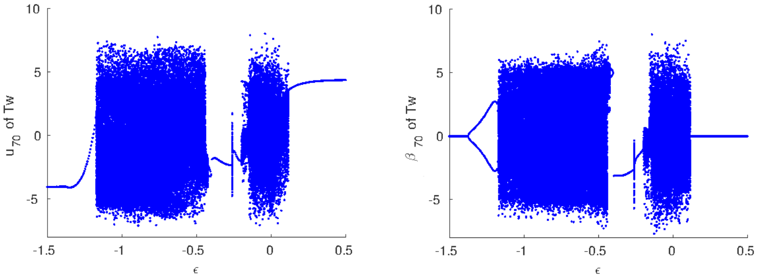


Chaotic attractor

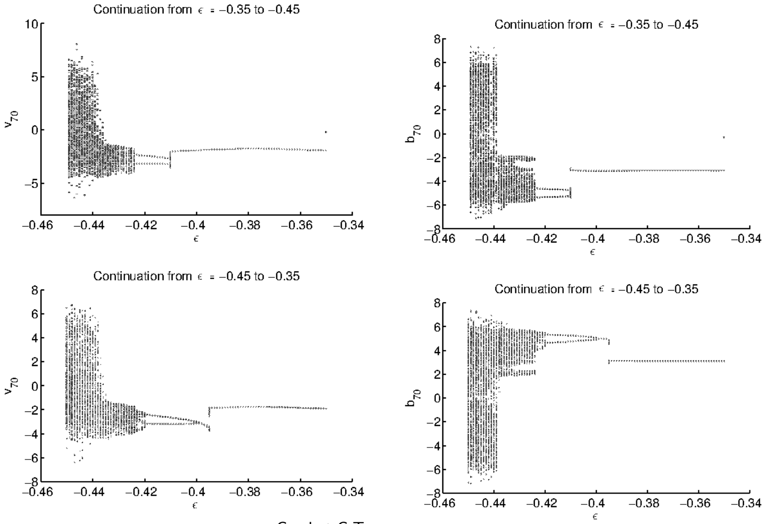




Bifurcation Diagram



Multistability



Asymptotic travelling wave solution

- Let us first consider the case $b_n = 0$. For τ sufficiently big,

$$u_n(\tau) = aU(n - a\tau) \tag{24}$$

- We define

$$y = \frac{1}{\log h} \int_0^{1/a} R(\tau) d\tau, \quad V(t - t_c) = \exp\left(\int_0^{\tau} R(\tau) d\tau \right) U(-\tau) \tag{25}$$

where τ is related to t by (16) and R is given by (20).

- **Theorem:** If $y > 0$, then solution $v_n(t)$ related to (24), for arbitrary posivite constant a, is given by

$$v_n(t) = ak_n^{y-1} V(ak_n^y (t - t_c)) \tag{26}$$

where the blowup time $t_c < \infty$ is given by

$$t_c = \int_0^{\infty} \exp\left(- \int_0^{\tau'} R(\tau'') d\tau'' \right) d\tau' \tag{27}$$

Proof schematics

- First, we prove that the limit t_c converges. Using the periodicity of R, from the definition of y

$$\int_0^\tau R(\tau')d\tau' > D + \tau y \log h. \tag{28}$$

Using the definition of t_c (27), for every positive y

$$t_c < \int_0^\infty \exp\left(-D - \tau y \log h\right) d\tau < \infty. \tag{29}$$

- Taking y and using $h_n = h^n$, from the periodicity of R

$$k_n^y = \exp\left(\int_\tau^{\tau+n} R(\tau')d\tau'\right). \tag{30}$$

- We study the solution at a time t' corresponding to $\tau' = \tau + n$,

$$t_c - t' = \int_{\tau+n}^{\infty} \exp\left(-\int_0^{\tau'} R(\tau'')d\tau''\right) d\tau' = k_n^{-y}(t_c - t). \tag{31}$$

- Then, from the limiting solution and the renormalization scheme,

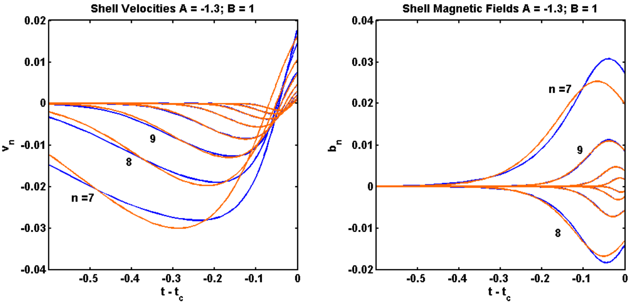
$$v_n(t') = k_n^{y-1} \exp\left(\int_0^{\tau} R(\tau')d\tau'\right) U(-\tau) = k_n^{y-1} V(t - t_c). \tag{32}$$

- Note that, (28) implies that

$$\exp\left(\int_0^{\tau} R(\tau')d\tau'\right) \to \infty \quad \text{as} \quad \tau \to \infty. \tag{33}$$

According to (17) and (24), this yields an unbounded norm $\|v'\|_\infty$ for $t \to t_c^-$.

Asymptotic Blowup Solution of Period (1,2)



Conclusions

- We prove an analytic criterion for blowup;
- Our method constructs asymptotic blowup solutions;
  - These solutions are universal: depend only on the attractor, selected by the value of an invariant;
  - We show that there is blowup in the case of existence of these attractors;
- Asymptotic solutions give scaling laws near blowup, useful for other applications;
- First (to our knowledge) observation of coexisting blowup scenarios.

Main References

[1] C.Gloaguen, J.Léorat, A.Pouquet, R.Grappin, A scalar model for MHD turbulence, *Physica D*, 17:154-182, 1985
[2] P.Constantin, B.Levant and E.S.Titi, Regularity of inviscid shell models of turbulence, *Phys. Rev. E*, 75(1):016304, 2007
[3] T.Dombre and J.L.Gilson, Intermittency, chaos and singular fluctuations in the mixed Obukhov-Novikov shell model of turbulence, *Physica D*, 111(1-4):265-287, 1998
[4] A.A. Mailybaev, Bifurcations of blowup in inviscid shell models of convective turbulence *Nonlinearity*, 26:1105-1124, 2013

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IMPA PRH/ANP-32

prh Programa de Recursos Humanos da ANP

anp Agência Nacional do Petróleo, Gás Natural e Biocombustíveis